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MEASUREMENT ERROR IN SEQUENTIAL TEST FOR THE NORMAL MEAN

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SUMMARY

Effect of measurement error on *OC* and *ASN* functions of the normal theory sequential test for mean is investigated. Numerical results are given to illustrate the mathematical findings.

Keywords: *OC* and *ASN* Functions ; Measurement error; Offset error; Sequential Test.

Introduction

It is well recognised by researchers and practitioners that, in general, the measurements are fallible. The effect of measurement error on certain statistical tests and acceptance sampling plans have been extensively studied by several workers. Some references may be made to Singh [3], Kakoty [1], Owen and Chou [2]. The Wald's sequential test under measurement error has, however, received no attention. The purpose of this paper is, therefore, to study through some numerical calculations, the effect of measurement error on *OC* and *ASN* functions associated with Wald's sequential probability ratio test (SPRT) for the mean of a normal distribution.

For the random variable X distributed normally with mean θ and known standard deviation σ, the *OC* and *ASN* functions of the SPRT with stopping bounds B and A due to Wald [4] for testing the hypothesis $H_0 : \theta = \theta_0$ versus $H_1 : \theta = \theta_1$ are given by

$$L(\theta) = \frac{A^{h(\theta)} - 1}{A^{h(\theta)} - B^{h(\theta)}} \tag{1.1}$$

and

$$ASN = E_\theta(n) = \frac{L(\theta) \log B + [1 - L(\theta)] \log A}{E_\theta(z)} \qquad (1.2)$$

respectively, where

$$z = (\theta_1 - \theta_0)(x - \bar{\theta})/\sigma^2 \qquad (1.3)$$

$$h(\theta) = 2(\bar{\theta} - \theta)/(\theta_1 - \theta_0) \qquad (1.4)$$

$$E_\theta(z) = (\theta_1 - \theta_0)(\theta - \bar{\theta})/\sigma^2 \qquad (1.5)$$

$$\bar{\theta} = (\theta_1 + \theta_0)/2$$

2. Effect of Measurement Error and Bias on OC and ASN Functions

In order to investigate the effect of measurement error on *OC* and *ASN* functions, we first consider the case when there is no offset error. It is assumed that the observed measurement Y and the true measurement X are linearly related i.e. $Y = X + e$, where e is the random error of measurement. It is further assumed that e is normally distributed with mean zero and variance σ_e^2 and is independent of X. The random variable Y will then be normally distributed with the same mean θ but a different variance σ_Y^2 $(= \sigma^2 + \sigma_e^2)$.

The approximate *OC* and *ASN* functions of the SPRT based on the observed sample $y_1, y_2, \ldots, y_n$ can be found to be

$$L'(\theta) = \frac{A^{h'(\theta)} - 1}{A^{h'(\theta)} - B^{h'(\theta)}},\ \theta \neq (\theta_1 + \theta_0)/2 \qquad (2.1)$$

and

$$E'_\theta(n) = \frac{L'(\theta) \log B + [1 - L'(\theta)] \log A}{E_\theta(z)},\ \theta \neq \frac{\theta_1 + \theta_0}{2} \qquad (2.2)$$

where $h'(\theta)$ is the non-zero solution of

$$E[e^{z'}]^{h'(\theta)} = 1 \qquad (2.3)$$

with $z' = (\theta_1 - \theta_0)(y - \bar{\theta})/\sigma^2$ and $E_\theta(z)$ as given in (1.5). Solving the equation (2.3), we have after some simplification

$$h'(\theta) = \rho^2 h(\theta); \qquad \rho = \sigma/\sigma_y \qquad (2.4)$$

where $h(\theta)$ given by (1.4) for infallible observations.

The actual errors of first and second kind will be given by

$$\alpha' = 1 - L'(\theta_0),\ \beta' = L'(\theta_1) \qquad (2.5)$$

and not by the nominal values of α and β. However, $\rho = 1$ corresponds to $\sigma_e = 0$, the case of no measurement error.

It should be further noted that at $\theta = \bar{\theta}$, the *OC* and *ASN* functions are given by

$$L'(\bar{\theta}) = -\log A/(\log B - \log A) \tag{2.6}$$

and

$$E'_{\theta}(n) = (-\rho^2 \sigma^2 \log A \cdot \log B)/(\theta_1 - \theta_0)^2 \tag{2.7}$$

So far our assumption has been that the measuring instrument is perfectly calibrated and there is no offset error or bias. We now generalise the model to include both the bias and the variability in measuring device by writing

$$Y = \theta_e + X + e \tag{2.8}$$

where θ_e is the mean bias.

Now, for the observed measurement Y, we have

$$E(Y) = \theta + \theta_e \quad \text{and } \sigma_Y^2 = \sigma_2 + \sigma_e^2 \tag{2.9}$$

The approximate *OC* and *ASN* functions in this case will retain the same form as (2.1) and (2.2) with

$$h'(\theta) = \frac{2(\bar{\theta} - \eta \cdot \theta)}{(\theta_1 - \theta_0)} \cdot \rho^2 \tag{2.10}$$

and

$$E'_{\theta}(z) = \frac{\theta_1 - \theta_0}{\sigma^2} (\eta \cdot \theta - \bar{\theta}); \eta = (\theta + \theta_e)/\theta \tag{2.11}$$

Here, it may be noted that $\eta = 1 = \rho$ $(\theta_e = 0 = \sigma_e)$ implies the absence of both the measurement error and the offset error; $\rho = 1$ $(\sigma_e = 0)$ the presence of only offset error and $\eta = 1$ $(\theta_e = 0)$ the presence of only measurement error.

3. Illustrative Numerical Comparisons

In order to illustrate the results found in the preceding section, we consider the following sequential test with $\sigma^2 = 1$.

$$H_0 : \theta = 3, H_1 : \theta = 4$$

To study the effect of measurement error on *OC* and *ASN* functions, we have considered the plan $\alpha = .05$, $\beta = .10$. The values of *OC* and *ASN* functions for some chosen values of θ and $\rho = 1, .8, .6$ have been worked out using equation (2.1) and (2.2). To give a visual comparison of *OC* and *ASN* functions, curves have been drawn in Fig. (1) and Fig. (2). It is seen from the Fig. (1) that measurement error seriously affects the *OC* curve of the normal theory sequential test for mean. The measurement

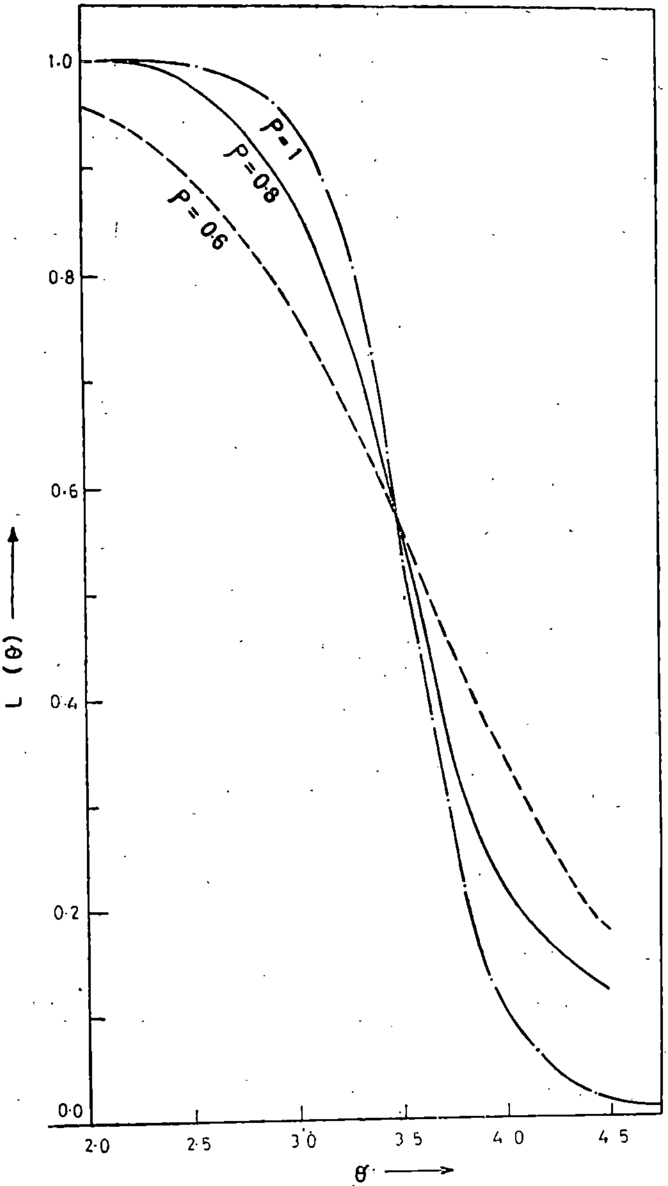

Fig. 1. *OC* Curves for different values of ρ with $\theta_0 = 3$, $\theta_1 = 4$, $\sigma = 1$, $\alpha = 0.05$, $\beta = 0.10$,

error is dependent on the amount of variability in the error term. As soon as this amount increases i.e. ρ decreases, the effect of measurement

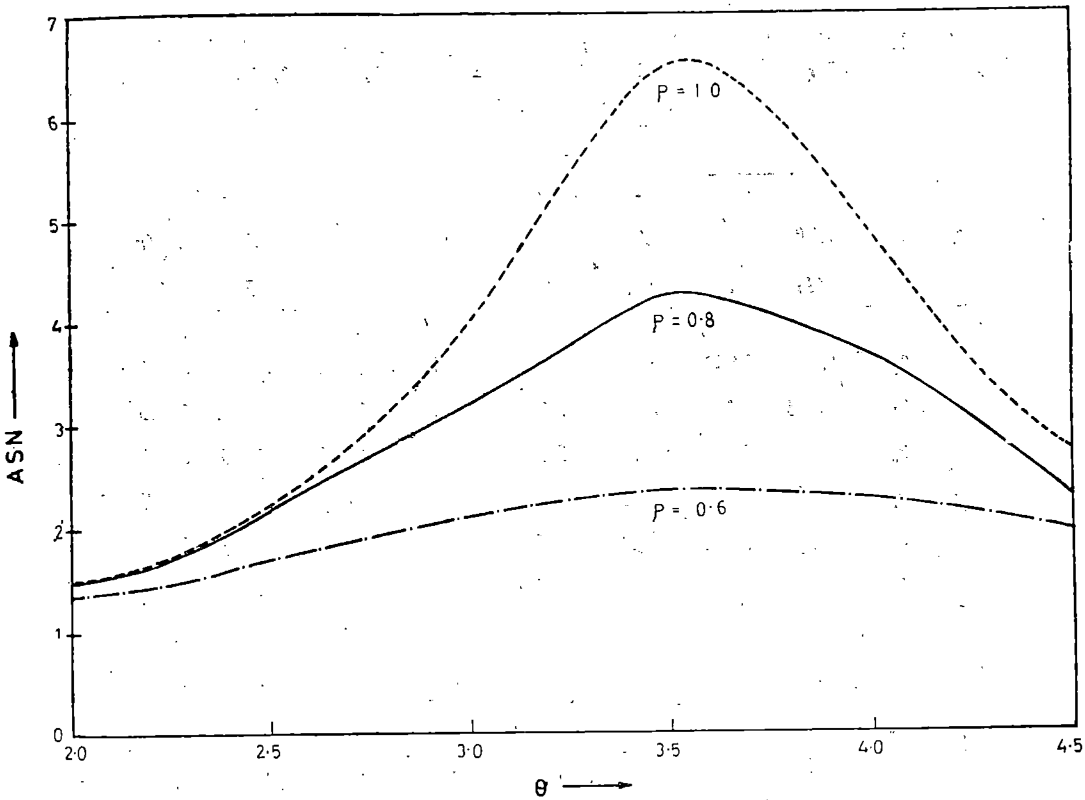

Fig. 2. *ASN* curves for different values of ρ with $\theta_0 = 3$, $\theta_1 = 4$, $\partial = 1$, $\alpha = 005$, $\beta = 0.10$.

error on *OC* function becomes more serious. For example, for the plan $\alpha = .05$, $\beta = .10$, at $\theta = 3.2$ the value of *OC* function for normal case ($\rho = 1$) is given by 0.863. This value decreases to 0.778 and 0.692 for $\rho = .8$ and .6 respectively. Similarly, at $\theta = 3.8$ the value of normal *OC* function (0.223) increases to 0.328 and 0.426 for $\rho = .8$ and .6 respectively. The above facts might be summarised by stating that the measurement error causes increase in both the producer's risk as well as the consumer's risk. The above phenomenon is also observed for the other combinations of α and β. From Fig. (2) for *ASN* functions, it is observed that the measurement error reduces the sample size, the reduction being in proportion to the size of the error.

For avoiding unwieldy growth of the section, the joint effect of bias and measurement error on *OC* and *ASN* functions has been studied only at two points viz. at the *AQL* and *LTPD*. Values of *OC* and *ASN* functions have been presented in Table 1 for different combinations of ρ and η.

TABLE 1—VALUE OF $L'(\theta)$ AND $E'_\theta(n)$ FOR SOME CHOSEN VALUES OF η AND ρ FOR THE CASE OF NORMAL POPULATION
($\theta_0 = 3, \theta_1 = 4, \sigma = 1$)
$\alpha = .05, \beta = .10$

ρ		= 1		= .8		= .6		= .4	
η	θ	$L'(\theta)$	$E'_\theta(n)$	$L'(\theta)$	$E'_\theta(n)$	$L'(\theta)$	$E'_\theta(n)$	$L'(\theta)$	$E'_\theta(n)$
		(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
.80	θ_0	1.000	2.047	.984	1.970	.914	1.646	.763	0.941
	θ_1	.863	5.155	.778	3.708	.692	2.233	.622	1.025
.85	θ_0	.996	2.348	.972	2.219	.888	1.763	.7396	0.960
	θ_1	.683	6.235	.641	4.072	.607	2.318	.582	1.037
.90	θ_0	.990	2.753	.953	2.513	.855	1.882	.714	0.979
	θ_1	.436	6.504	.481	4.184	.516	2.354	.542	1.044
.95	θ_0	.978	3.289	.923	2.851	.815	1.999	.688	.995
	θ_1	.224	5.804	.328	4.014	.426	2.337	.501	1.047
1.00	θ_0	.950	3.988	.875	3.221	.767	2.109	.660	1.010
	θ_1	.100	4.752	.207	3.650	.341	2.272	.461	1.044
1.05	θ_0	.892	4.848	.807	3.592	.712	2.205	.632	1.022
	θ_1	.042	3.820	.123	3.225	.267	2.171	.421	1.038
1.10	θ_0	.786	5.749	.714	3.915	.651	2.281	.602	1.032
	θ_1	.017	3.113	.072	2.799	.204	2.046	.382	1.027
1.15	θ_0	.624	6.403	.602	4.127	.585	2.332	.572	1.039
	θ_1	.007	2.304	.041	2.434	.154	1.910	.346	1.012

A comparison of the values of $L'(\theta)$ for error free case ($\rho = 1 = \eta$) with other entries of column (1) clearly shows that in the absence of variability in measurement error ($\rho = 1$) producer gains undue protection due to negative bias whereas consumer loses. Reverse is the situation for positive bias. Similar effect of bias on producer's and consumer's risk is observed even if there is variability in measurement error ($\rho \neq 1$). The values of *ASN* function given in the Table reveal that bias seriously affects the average sample number irrespective of the magnitude of the error variance. The maximum effect can be studied only after drawing the entire *ASN* curve.

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REFERENCES

[1] Kakoty, S. (1982) : A note on sensitivity of the ANOVA tests when observations are subject to error, *Journal of Indian Society of Agricultural Statistics*, 34 (3) : 75-78.

[2] Owen, D. B. and Chou, Youn-Min (1933) : Effect of measurement error and instrument bias on operating characteristics for variables sampling plans, *Journal of Quality Technology*, 15 (3) : 107-117.

[3] Singh, H. R. (1966) : Test and measurement error in quality control, *Unpub. Ph.D. Thesis*, IIT, Kharagpur.

[4] Wald, A. (1947) : *Sequential Analysis*, John Wiley and Sons, New York.